Critical transition in the constrained traveling salesman problem

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We investigate the finite size scaling of the mean optimal tour length as a function of density of obstacles in a constrained variant of the traveling salesman problem (TSP). The computational experience pointed out a critical transition (at $\rho_c \approx 85\%$) in the dependence between the excess of the mean optimal tour length over the Held-Karp lower bound and the density of obstacles.

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The traveling salesman problem (TSP) consists of finding the length of the shortest closed tour visiting N cities. TSP is one of the most widely studied combinatorial optimization problems [1,2]. TSP is a classic NP-complete problem, i.e., no algorithm exists for solving the problem in polynomial time. TSP is also one of the few combinatorial optimization problems that have been studied extensively in the context of statistical physics [3–7].

The importance of this problem is due to its formulation simplicity coupled with its intractability, which has lead to the development of a variety of general algorithms for dealing with complex optimization problems. The amount of calculation required for finding the exact solution of an NP-complete problem increases exponentially with problem size. To cope with this difficulty, global optimization schemes can be used to find reasonable solutions of these problems. Stochastic methods like simulating annealing provide a probabilistic guarantee of convergence upon a global minimum [7]. Formally, the symmetric Euclidean TSP is defined as follows: given a set of distinct points $\{p_i=(x_i,y_i)\in[0,1]^2, i=1,...,N\}$, we find a permutation $\pi=\{\pi(1),\ldots,\pi(N)\}$ of $P=\{1,...,N\}$ that minimizes the tour length

$$L(\pi)=\sum_{i=1}^{N} d(p_{\pi(i)},p_{\pi[(i+1)\mathrm{mod}(N)]}), \tag{1}$$

where

$$d(p_{\pi(i)},p_{\pi(j)})=\sqrt{(x_{\pi(i)}-x_{\pi(j)})^2+(y_{\pi(i)}-y_{\pi(j)})^2} \tag{2}$$

is the Euclidean distance between a pair of points.

It has been shown [8] that in the large-N limit the optimal tour length for a given instance L_{opt} is self-averaging up to a scaling factor

$$\lim_{N\to\infty}\frac{L_{\mathrm{opt}}}{\sqrt{N}}=\beta, \tag{3}$$

where convergence to the instance-independent β is with probability 1 in the ensemble of instances with randomly distributed cities. Recent computational experiments suggest this constant is $\beta=0.7124$ [5,9].

TSP can also be formulated as a linear program based on the interpretation of the optimal tour as a minimum-length Hamiltonian circuit in the complete graph with cities as vertices [10]. For instances of moderate size the solution to this problem can be computed exactly using linear programming. However, the method is not suitable for large N because the number of constraints induced in the linear program is exponential in N. The exact solution will be the lower bound of the optimal tour length, called the Held-Karp lower bound [10]. By using state-of-the-art computers, combined with the linear programming method and statistical averages over a large number of random Euclidean instances, the expected ratio of the Held-Karp (HK) lower bound to $\sqrt{N}$ has been calculated [9] for moderate values of N and it is given by

$$C_{\mathrm{HK}}(N)=0.708\,05+0.522\,29N^{-0.5}+1.315\,72N^{-1}-3.074\,74N^{-1.5}. \tag{4}$$

The error in the above formula is less than 0.1%. Obviously, the expected Held-Karp lower bound to $\sqrt{N}$ is approaching a constant $C<\beta$ as $N\to\infty$.

The Held-Karp constant $C_{\mathrm{HK}}(N)$ has become the standard test for evaluating the performance of various heuristic algorithms proposed to solve the TSP. The practical computational strategy consists of evaluating the average of the heuristic optimal tour length, $\langle L_{\mathrm{opt}}(N)\rangle$, for a large number of N-city random Euclidean instances, and to compute the excess over the Held-Karp lower bound, as follows:

$$\varepsilon(N)=\left[\frac{\langle L_{\mathrm{opt}}(N)\rangle N^{-0.5}}{C_{\mathrm{HK}}(N)}-1\right]\times 100\%. \tag{5}$$

In this work we study a constrained variant of TSP. The constrained model considered here is mainly motivated by the obstacle avoidance problem in robot navigation. The task of the robot is to find the shortest closed tour visiting N cities (goals) in an environment with obstacles.

Formally, the constrained TSP can be defined as follows: given a set of obstacles

$$\{O_k\subset[0,1]^2, \quad k=1,...,M\}$$

and a set of distinct points (cities, goals)

$$\left\{p_i=(x_i,y_i)\in[0,1]^2-\bigcup_{k=1}^{M}O_k, \quad i=1,...,N\right\},$$

we find a permutation $\pi=\{\pi(1),\ldots,\pi(N)\}$ of $P=\{1,...,N\}$ that minimizes the closed tour length visiting all the cities. In the above definition, the constraints are introduced by the

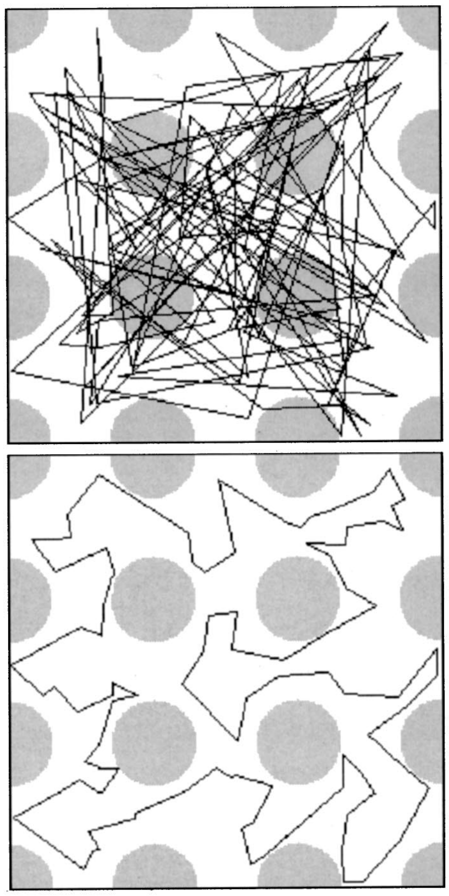

FIG. 1. A constrained TSP instance with 100 cities.

intersections between the segments connecting distinct pairs of cities and the obstacles. Obviously, these intersections are not allowed and they must be penalized somehow. The constrained TSP cost function can be written as follows:

$$L(\pi)=\sum_{i=1}^{N}[d(p_{\pi(i)},p_{\pi[(i+1)\mathrm{mod}(N)]})+c(p_{\pi(i)},p_{\pi[(i+1)\mathrm{mod}(N)]})], \qquad (6)$$

where the supplementary costs, or the penalties, are defined by

$$c(p_{\pi(i)},p_{\pi(j)})=\begin{cases}0 & \text{if } \sigma(\pi(i),\pi(j))\cap(\cup_{k=1}^{M}O_k)=\varnothing \\ w & \text{otherwise}\end{cases}. \qquad (7)$$

Here $\sigma(\pi(i),\pi(j))$ is the segment connecting the pair of cities $(\pi(i),\pi(j))$ and $w>0$ is the constant penalty introduced when this segment intersects at least one obstacle.

An example of constrained TSP is given in Fig. 1. Here, the obstacles correspond to an array structure of circular objects (disks) and an intersection was penalized with $w=\sqrt{2}$. Let O_k be a circular obstacle at position (x_k,y_k) with the radius R and let $\sigma(\pi(i),\pi(j))$ be the segment connecting the points $p_{\pi(i)}=(x_{\pi(i)},y_{\pi(j)})$ and $p_{\pi(j)}=(x_{\pi(j)},y_{\pi(j)})$, respectively. The line containing the segment σ has intersected the circular object O_k if the smallest distance from (x_k,y_k) to the line is less than R. This is calculated in two steps. First we try to trivially reject the intersection by checking if the rectangle spanned by the segment intersects with the circular obstacle; if it does not, we know that there is no intersection. If the rectangle and the circle intersect, we have to check the distance between the line and the center of O_k to see if it is less than the radius of O_k. If it is, we have an intersection.

From the above considerations it follows that the constrained TSP requires a significant increase of the computation effort compared to the standard TSP. However, this effort is substantially reduced if the matrices $d=[d_{ij}]$ and $c=[c_{ij}]$ are calculated at the beginning of the algorithm.

The best heuristics for TSP are *local search* [11] and *simulated annealing* [7]. These algorithms typically get about 1% of the optimal tour length [2]. These successes leave less room for the development of other new heuristic algorithms. For our numerical simulation purposes, we have used a combination of these two approaches.

Simulated annealing (SA) is a general method of optimization. In case of TSP, one starts with a ''solution'' given by the nearest-neighbor algorithm. A new solution is constructed by imposing a random permutation of cities. If the cost of a new state (the length of the tour) is lower than that of the previous one, the change is accepted unconditionally and the system is updated. If the cost is greater, the new solution is accepted with the probability $\exp(-\Delta L/T)$. This is the Monte Carlo step, the fundamental procedure of SA. Repeated applications of the Monte Carlo step result in a Boltzmann distribution of microstates. As the temperature parameter T is decreased, this procedure allows the system to move consistently towards lower cost states, yet still ''jump'' out of local minima due to the occasional acceptance of an upward move. If the temperature is decreased logarithmically, SA guarantees an optimal solution with a nonzero probability. However, a logarithmic schedule is quite slow and in practice one uses cooling schedules that drop the temperatures much more rapidly as γ^t, where $0<\gamma<1$ and t is the time step. This can be obtained, for instance, by performing a fixed number of trials at each temperature, after which one declares ''equilibrium'' and reduces the temperature by a standard factor γ. Under such an exponential cooling regime, the temperature will after a polynomially bounded amount of time reach values sufficiently close to zero that uphill moves will no longer be accepted and one can declare ''freezing'' to have set in. With such a polynomially bounded cooling schedule, SA is only an approximation algorithm.

We have used an exponential cooling schedule with the initial temperature given by $T_0=1.5N^{-0.5}$ as suggested in [2]. The final temperature was fixed to $T_f=0.001$ and the total number of temperatures was also fixed to $M=25$. It follows that the discount factor is given by $\gamma=(T_f/T_0)^{1/M}$. The temperature length (the number of steps at a given temperature) was calculated using the following simple adapting scheme: $\alpha N(N-1)(2-T/T_0)$.

In our SA implementation the ''new solutions'' were generated by using a random (equiprobable) selection between the 2-Opt and 3-Opt moves such as in the local-optimization

TABLE I. Excess over the Held-Karp lower bound for SA algorithm.

N	$\langle\varepsilon\rangle$ (%) ($\alpha=1$)	$\langle\varepsilon\rangle$ (%) ($\alpha=10$)	$\langle\varepsilon\rangle$ (%) ($\alpha=10^2$)
10^2	2.51	1.58	0.95
5×10^2	2.84	1.83	1.17
10^3	3.17	2.06	1.25

heuristics. The 2-Opt move deletes two edges, thus breaking the tour into two paths, and then reconnects those paths in the other possible way. In 3-Opt moves, the exchange replaces up to three edges of the current tour (see [1,2] and [11] for details). In Table I we give the excess over the Held-Karp lower bound (5) for our SA implementation.

We have applied the above SA algorithm in the case of constrained TSP by computing the dependence between the excess of the mean optimal tour length over the Held-Karp lower bound and the density of obstacles. Because of the larger CPU time consuming in constrained TSP and because we need to generate good statistical results, there is necessarily a tradeoff in the choice of N and in the method used for generating obstacles. We have chosen an average value, $N=10^2$, for which we were able to generate 10^2 constrained TSP instances for each density of obstacles. The obstacles density was simulated by introducing random (symmetrical) penalties in the cost function (6). For a given density ρ, the number of randomly introduced penalties was computed as $\rho N(N-1)/2$. We have chosen a high value for the penalty, $w=\sqrt{2}N$, in order to see exactly where the critical transition is. This means N times the maximum possible distance in the unit square.

The results of the computation are given in Fig. 2. One can see the sharp transition starting at $\rho_c\approx85\%$ in the dependence between the excess of the mean optimal tour length over the Held-Karp lower bound and the density of obstacles. We have obtained the same result for $N=n\times10^2$ $(n=2,3,...10)$. In this case we have to check just for $\rho=80,81,...,90\,\%$ and ten constrained TSP instances for each density of obstacles. This result shows that the obtained critical density is not depending on N.

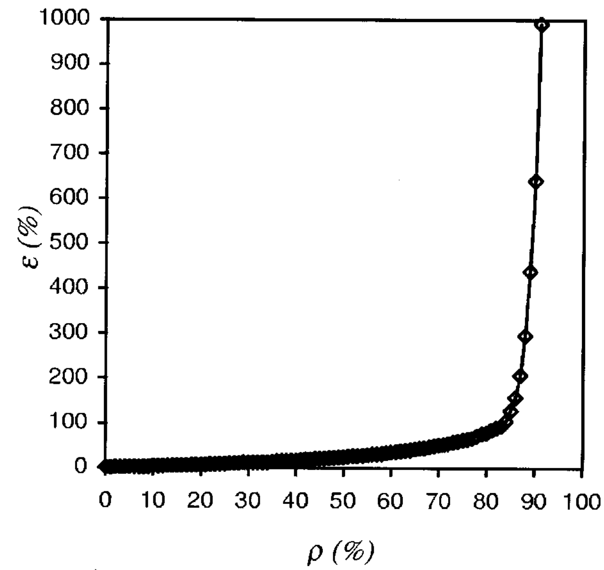

FIG. 2. The dependence between the excess of the mean optimal tour length over the Held-Karp lower bound and the density of obstacles.

All TSP instances have been generated and solved using an independent parallelism strategy, i.e., using different initial conditions and different random streams on each processor (Compaq Alpha cluster, C++/TruUnix64). For densities larger than the critical value $\rho>\rho_c$ the algorithm is not capable of finding the shortest tour and large penalties are introduced in the cost function. However, this critical value is pretty high and the robot will be able to find a shortest closed tour even in a very cluttered environment.

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[1] *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, edited by E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy, and D. B. Shmoys (Wiley, New York, 1985).
[2] *Local Search in Combinatorial Optimization*, edited by E. H. L. Aarts and J. K. Lenstra (Wiley, New York, 1997).
[3] M. Mezard and G. Parisi, Europhys. Lett. **2**, 913 (1986).
[4] W. Krauth and M. Mezard, Europhys. Lett. **8**, 213 (1989).
[5] A. G. Percus and O. C. Martin, Phys. Rev. Lett. **76**, 8 (1996); **78**, 1188 (1996).
[6] N. J. Cerf, J. Boutet de Monvel, O. Bohigas, O. C. Martin, and A. G. Percus, J. Phys. I **7**, 117 (1997).
[7] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi, Science **220**, 671 (1983).
[8] J. Beardwwod, J. H. Halton, and J. M. Hammersley, Proc. Cambridge Philos. Soc. **55**, 299 (1959).
[9] D. S. Johnson, L. A. McGeoch, E. E. Rothberg (unpublished).
[10] M. Held and R. M. Karp, Oper. Res. **18**, 1138 (1970).
[11] S. Lin and B. W. Kernighan, Oper. Res. **21**, 498 (1973).